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Size Distribution of Fractured Areas in One-Dimensional Systems

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We study a one-dimensional model for fracture, identifying fractured areas with intervals on which a stress field ξ exceeds a threshold value Δ. When ξ is a diffusion process, the cumulative number $N(l)$ of fractured areas whose length is greater than l obeys a power law Cl^{-p} as $l \downarrow 0$ with probability one. The exponent p and the constant C are determined. The exponent p agrees with the Hausdorff dimension of the end points of fractured areas, i.e., $\xi^{-1}(\Delta)$. Even if ξ is self-similar with parameter $H>0$, i.e., $\xi(cx)-\Delta$ is equivalent to $c^H\{\xi(x)-\Delta\}$ for any $c>0$, the exponent p does not depend solely on H; $p=\lambda H$, where $\lambda \in (0, 1/H)$ is another parameter characterizing ξ. Non-diffusion processes are given where $N(l)$ does not follow a power law.

KEY WORDS: Fracture; size distribution; power law; diffusion process; Hausdorff dimension; self-similar.

1. INTRODUCTION AND SUMMARY

When inhomogeneous materials such as rocks are compressed, many small fractures occur, emitting elastic waves: so-called acoustic emission. The cumulative number $N(E)$ of fractures whose emitted energy is greater than E obeys a power law distribution

$$N(E) \sim C_0 E^{-p} \tag{1.1}$$

The relation (1.1) remains valid for much larger fractures, i.e., earthquakes. It is called the Gutenberg–Richter or the Ishimoto–Iida relation and plays an important role in seismology.[1] If we assume that the released

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energy of each fracture is proportional to its size S, a power-law distribution (1.1) with different C_0 also holds for S.

Stimulated by the fractal theory,[2] it has been pointed out that the power law for S readily follows from the assumption of self-similarity.[3] The assumption is naturally inferred from the fractal surfaces of fractured materials[4] or the resemblance of fracture to a phase transition.[5] However, it would be hasty to suppose that the self-similarity alone is responsible for the power law. We present here a model that is not necessarily self-similar but has a power-law size distribution. Even when the model becomes self-similar, the exponent p does not depend solely on the self-similarity parameter H [see (1.10) for definition].

The model we study here was originally proposed by Oda *et al.*[6] Let $\sigma_{ij}(x)$ be a random stress field of the material. A fractured area will be defined as a connected region fulfilling a fracture criterion

$$\xi(x) = G(\sigma_{ij}(x)) \geqslant \Delta \tag{1.2}$$

where G is a certain scalar function and Δ is a threshold value for fracture. (See Fig. 1.) If we adopt, for example, von Mieses' criterion, $G(\sigma_{ij})$ is the so-called equivalent stress, given explicitly as[7]

$$\begin{aligned} G(\sigma_{ij}) = 2^{-1/2}[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 \\ + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)]^{1/2} \end{aligned} \tag{1.3}$$

The size distribution for fractured areas can be obtained in principle by solving a stochastic equation governing σ and invoking (1.2). We avoid this tedious process; we impose some conditions on ξ itself and apply (1.2).

In the following we restrict ourselves to a one-dimensional material with length L. By $N(l, L)$ we denote the cumulative number of fractured areas whose length is greater than l.

As we see in Section 4, $N(l, L)$ does not necessarily follow a power law as in (1.1). One well-known example that exhibits a power law is Brownian motion B; the relation

$$N(l, L) \sim C_1 l^{-1/2} \qquad \text{as} \quad l \downarrow 0 \tag{1.4}$$

holds with probability one, where C_1 is a constant depending on the sample parameter and L. (See Ref. 8, Section 2.2, and this paper, Section 3.) The Brownian motion, however, has two special properties—the Gaussian property and self-similarity. So we are naturally led to the following questions: (1) Does $N(l, L)$ follow a power law even if the process ξ lacks self-similarity or the Gaussian property? (2) If it does, how is the exponent p given? (3) How does p depend on the similarity parameter H if ξ is self-

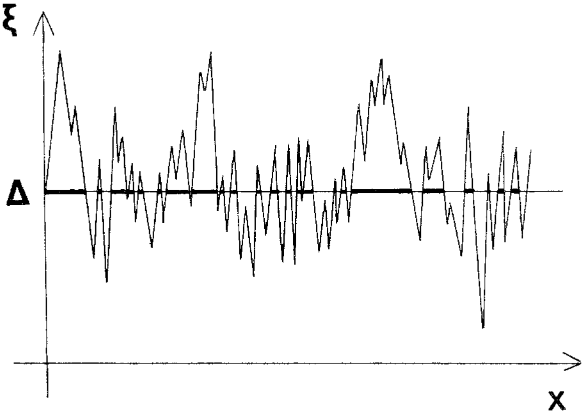

Fig. 1. Schematic illustration of fracture. Bold segments are fractured areas.

similar? (4) How is p related to the Hausdorff dimension of the surface of fractured materials (a problem heuristically discussed by Aki[9])?

Section 2 is devoted to solving these questions rigorously when ξ is a diffusion process. The result is as follows. The process ξ is uniquely characterized by two functions s and m in the sense that the Kolmogorov backward operator A is given by

$$A = \frac{d}{dm}\frac{d}{ds} \tag{1.5}$$

Here s and m are usually called the scale and the speed measure, respectively. We assume, without losing generality in practical situations, that they have the following asymptotic forms around the threshold Δ in (1.2):

$$s(r) \sim C_2(r-\Delta)^{\lambda}, \qquad m(r) \sim C_3(r-\Delta)^{\mu} \tag{1.6}$$

as $r \downarrow \Delta$, where C_2, C_3, λ, and μ are positive constants. Then $N(l, L)$ follows a power law,

$$N(l, L) \sim Cl^{-p} \qquad \text{as} \quad l \downarrow 0, \quad p = \lambda/(\lambda+\mu) \tag{1.7}$$

with probability one. The exponent p agrees with the Hausdorff dimension of end points of fractured areas

$$\mathscr{Z}_\Delta = \{0 \leqslant x \leqslant L: \xi(x) = \Delta\} \tag{1.8}$$

The constant C is given by

$$C = C_4\phi(L) \tag{1.9}$$

where C_4 is defined by (2.16), and the random variable $\phi(L)$ is what probabilists call the local time given by (2.13) with $r=0$.

Two examples are discussed in Section 3. One is the Ornstein-Uhlenbeck process, for which $\lambda=\mu=1$, hence $p=1/2$. The other is a self-similar diffusion process fulfilling a self-similarity condition with parameter $H>0$

$$\xi(cx)-\Delta \stackrel{d}{=} c^H\{\xi(x)-\Delta\} \tag{1.10}$$

for any $c>0$. Here the symbol $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions of the two processes, or, in physics terminology, agreement of all multipoint correlation functions of the two processes. Such a self-similar process is constructed when

$$s(r)=\begin{cases} C_5\,|r-\Delta|^{\lambda}, & r\geqslant\Delta, \\ -C_5'\,|r-\Delta|^{\lambda}, & r<\Delta, \end{cases} \qquad m(r)=\begin{cases} C_6\,|r-\Delta|^{-\lambda+1/H}, & r\geqslant\Delta \\ -C_6'\,|r-\Delta|^{-\lambda+1/H}, & r<\Delta \end{cases} \tag{1.11}$$

where C_5, C_5', C_6, and C_6' are positive constants and $\lambda\in(0, 1/H)$. In this case we have $p=\lambda H$, showing that the exponent p depends not only on H, but also on λ.

In Section 4, we discuss Gaussian processes having smooth sample functions. A Gaussian process ξ with mean Δ and covariance

$$E[\xi(x)\,\xi(y)]=xy+\Delta^2 \tag{1.12}$$

has only a finite number of fractures, i.e., $N(0+, L)<\infty$ for each $L<\infty$ with probability one. This fact invalidates the power law (1.7). The same is true for a stationary Gaussian process η with mean zero and covariance

$$E[\eta(x)\,\eta(y)]=\exp(-C_7\,|y|^2), \qquad C_7>0 \tag{1.13}$$

whose $N(l, L)$ was studied by simulation.[6] The process η has another interesting distribution; the cumulative number per unit length $N(l, L)/L$ converges as $L\to\infty$ to a certain distribution $\bar{N}(l)$ with probability one. If Δ is sufficiently large, $\bar{N}(l)$ scaled by $\bar{N}(0+)$, the average number of fractures per unit length behaves as

$$\bar{N}(l)/\bar{N}(0+)\sim\exp(-l^2C_7\Delta^2/4) \tag{1.14}$$

and does not obey a power law.

A sketch is given in Section 5 of the results by Kesten,[17] who studied the asymptotic behavior of $N(l, L)$ as $l\downarrow 0$ for stable symmetric processes. A stable symmetric process ξ with index H $(1/2\leqslant H<\infty)$ is a stochastic process with stationary independent increments whose characteristic function is given by

$$E(\exp\{i\theta[\xi(x)-\xi(y)]\}) = \exp(-|x-y|\,|\theta|^{1/H}) \tag{1.15}$$

It has, in contrast with the processes in Sections 2–4, discontinuous sample functions if $H > 1/2$ (the process with $H = 1/2$ is essentially the Brownian motion). Roughly speaking, $N(l, L)$ follows a power law

$$N(l, L) \sim l^{-p}, \qquad p = \max(H-1, 0) \tag{1.16}$$

but its precise implication given by Kesten is different from (1.7).

2. POWER LAWS FOR DIFFUSION PROCESSES

Let $\xi(x)$, $x \in [0, \infty)$, be an R^1-valued homogeneous diffusion process. Redefine the origin of ξ so that the fracture criterion is given by (1.2) with $\Delta = 0$. Let $\mathscr{X}^+$ be a fractured area, i.e., maximal open interval on which $\xi(x) > 0$. By $|\mathscr{X}^+|$ we denote the length (Lebesgue measure) of $\mathscr{X}^+$. Our concern is the asymptotic behavior as $l \downarrow 0$ of the cumulative number $N(l, L)$ of $\mathscr{X}^+$, contained in the interval $[0, L)$, whose length is greater than l:

$$N(l, L) = \#\{\mathscr{X}^+ : \mathscr{X}^+ \subset [0, L), |\mathscr{X}^+| \geqslant l\} \tag{2.1}$$

Let us specify conditions imposed on ξ from a physical point of view. Naturally ξ is conservative:

$$P_r(\xi(x) \in R^1, \forall x \in [0, \infty)) = 1, \qquad \forall r \in R^1 \tag{2.2}$$

where P_r denotes the probability for paths starting at r, i.e., $\xi(0) = r$. A point $r \in R^1$ is called regular if

$$P_r(\tau_{r+} = \tau_{r-} = 0) > 0 \tag{2.3}$$

where τ_u is the first hitting time for u

$$\tau_u = \inf\{x > 0, \xi(x) = u\} \tag{2.4}$$

and τ_{r+} $(\tau_{r-}) = \lim_{u \downarrow r} \tau_u$ (resp. $\lim_{u \uparrow r} \tau_u$). All values of the stress field ξ will be regular points. So we assume that the range of ξ is contained in an interval $I = (r_1, r_2)$ $(r_1 < 0 < r_2)$ consisting only of regular points. Behavior near the boundaries r_i $(i = 1, 2)$ is assumed as follows: r_i is inaccessible for ξ (natural or entrance boundary), or reflected as soon as ξ attains to r_i (regular boundary with instantaneous reflection). We further assume that ξ

is persistent (recurrent); for any $a, b \in I$, if $\xi(x) = a$, then $\xi(y) = b$ for some $y \in (x, \infty)$ with probability one. This may be intuitively understood that the stress ξ takes all the values in I many times so long as the length L is sufficiently large.

The nonsingular diffusion process ξ on the interval I is uniquely characterized by two quantities, a continuous, strictly increasing function s called the canonical scale and a measure dm called the speed measure. The process is determined by the generator

$$Af(r) \equiv \lim_{x \downarrow 0} \frac{E_r[f(\xi(x))] - f(r)}{x} = \frac{d}{dm} \frac{d^+}{ds} f(r) \tag{2.5}$$

together with a Neumann-type boundary condition at r_1 (r_2)

$$(d^+/ds)\, f(r_1) = 0 \tag{2.6}$$

$$[\text{resp. } (d^-/ds)\, f(r_2) = 0] \tag{2.7}$$

if r_1 (resp. r_2) is a regular boundary with instantaneous reflection. Here $E_r[\cdot]$ denotes the expectation with respect to P_r, and d^+/ds and d^-/ds are one-sided scale derivatives:

$$\begin{aligned} (d^+/ds)\, f(r) &= \lim_{h \downarrow 0} [f(r+h) - f(r)]/[s(r+h) - s(r)] \\ (d^-/ds)\, f(r) &= \lim_{h \downarrow 0} [f(r-h) - f(r)]/[s(r-h) - s(r)] \end{aligned} \tag{2.8}$$

The properties assumed above are given in terms of m and s. Put

$$\begin{aligned} A_1 &= \iint_{r_1 < v < u < 0} dm(u)\, ds(v) \\ A_2 &= \iint_{r_2 > v > u > 0} dm(u)\, ds(v) \\ B_1 &= \iint_{r_1 < v < u < 0} ds(u)\, dm(v) \\ B_2 &= \iint_{r_2 > v > u > 0} ds(u)\, dm(v) \end{aligned} \tag{2.9}$$

then the boundary r_i ($i = 1, 2$) is regular if $A_i < \infty$, $B_i < \infty$, exit if $A_i < \infty$, $B_i = \infty$, entrance if $A_i = \infty$, $B_i < \infty$, and natural if $A_i = \infty$, $B_i = \infty$; ξ is conservative and persistent iff

$$(1)\ s(r_1) = -\infty \quad \text{or} \quad (2)\ s(r_1) > -\infty, \quad m(r_1) > -\infty, \quad \text{and (2.6)} \tag{2.10}$$

and

$$(3)\ s(r_2)=\infty \qquad \text{or} \quad (4)\ s(r_2)<\infty, \qquad m(r_2)<\infty, \qquad \text{and (2.7)} \tag{2.11}$$

In (2.10), (2.11), m is the right-continuous, nondecreasing function defined by the relation

$$m(r)=\begin{cases}\int_{[0,r]} dm & (r\geq 0)\\ -\int_{[r,0)} dm & (r<0)\end{cases} \tag{2.12}$$

Without loss of generality we have put $m(0-)=0$ in (2.12) and will assume $s(0)=0$ in the following. We will identify m with dm when there is no chance of confusion. The quantity

$$\phi(L, r)=\text{measure}\{y\colon \xi(y)\in dr, 0\leqslant y\leqslant L\}/dm(r) \tag{2.13}$$

exists with probability one, and is called the local time. We write $\phi(L)$ for $\phi(L, 0)$. (See Itô and McKean[8] and Itô[10] for basic notions and results of diffusion processes.)

Now we give our main theorem:

Theorem. Suppose the scale s and the speed measure dm have the asymptotic forms as $r\downarrow 0$

$$s(r)\sim C_2 r^{\lambda} \qquad \text{and} \qquad m(r)\sim C_3 r^{\mu} \tag{2.14}$$

where C_2, C_3, λ, and μ are positive constants. Then

$$P_0(\lim_{l\downarrow 0} N(l, L)/C_4 l^{-p}=\phi(L), L>0)=1 \tag{2.15}$$

where the constant C_4 is given by

$$C_4=C_2^{p-1}C_3^{p}[(1-p)p]^{-p}\,\Gamma(1+p)^{-1} \tag{2.16}$$

and

$$p=\lambda/(\lambda+\mu) \tag{2.17}$$

In (2.16), Γ is the gamma function.

Remark 1. We need to assume the asymptotic form (2.14) only for $r>0$. This comes from the fact that the behavior of ξ above the level Δ $(=0)$ is determined just by s and m for $r>0$.

Remark 2. The exponent p agrees with the Hausdorff dimension of

$$\mathscr{Z}_0 = \xi^{-1}(0) = \{0 \leqslant x \leqslant L: \xi(x) = 0\} \tag{2.18}$$

which is given in Ref. 8, Section 6.7.

Proof of the Theorem. Let us start with the following result.

Lemma 1. (Ref. 8, Sections 6.2 and 6.3.) For a nonsingular and persistent diffusion process ξ,

$$P_0(\lim_{l \downarrow 0} N(l, L)/n_+[l, \infty) = \phi(L), L > 0) = 1$$

Here $n_+[l, \infty)$ is given by the monotonically increasing limit of $P_r(\tau_0 \geqslant l)/s(r)$ as $r \downarrow 0$, i.e.,

$$P_r(\tau_0 \geqslant l)/s(r) \uparrow n_+[l, \infty) \qquad \text{as} \quad r \downarrow 0 \tag{2.19}$$

and is continuous in l.

From this lemma we only have to study the asymptotic behavior of $n_+[l, \infty)$ as $l \downarrow 0$. Let us consider the differential equation on I:

$$\frac{d}{dm} \frac{d^+}{ds} g(r) = \alpha g(r), \qquad \alpha > 0 \tag{2.20}$$

It is known[10] that the special solutions e_0, e_1 determined by

$$e_0(r) = 1 + \alpha \int_0^r \int_{0+}^{v+} e_0(u)\, dm(u)\, ds(v) \tag{2.21}$$

$$e_1(r) = s(r) + \alpha \int_0^r \int_{0+}^{v+} e_1(u)\, dm(u)\, ds(v) \tag{2.22}$$

span the solutions of (2.20). Its positive and decreasing solution g_2 with $g_2(0) = 1$, and with the additional condition

$$(d^-/ds)\, g_2(r_2) = 0 \tag{2.23}$$

if r_2 is a regular boundary with instantaneous reflection, is uniquely given by

$$g_2(r) = e_0(r) - h(\alpha)^{-1}\, e_1(r) \tag{2.24}$$

where

$$h(\alpha) = \lim_{r \uparrow r_2} e_1(r)/e_0(r) \tag{2.25}$$

On the other hand, g_2 also is expressed in terms of the hitting time τ_0 as [Ref. 8, Section 4.6, p. 129, Eq. (3b); the points 0, 1/2, 1 there correspond to r_1, 0, r_2 in our case]

$$g_2(r)/g_2(0) = E_r[\exp(-\alpha\tau_0)], \qquad r > 0 \tag{2.26}$$

Using this expression, we have

$$\begin{aligned}\frac{1}{g_2(0)}\frac{d^+g_2}{ds}(0) &= \lim_{r\downarrow 0} s(r)^{-1}[g_2(r) - g_2(0)]\, g_2(0)^{-1} \\ &= \lim_{r\downarrow 0} s(r)^{-1}\left[\int_0^\infty e^{-\alpha l}P_r(\tau_0 \in dl) - 1\right] \\ &= \lim_{r\downarrow 0} s(r)^{-1}\left[\int_0^\infty P_r(\tau_0 > l)\, d(e^{-\alpha l} - 1)\right]\end{aligned}$$

By (2.19) and continuity of $n_+[l, \infty)$, we obtain the relation

$$\frac{1}{g_2(0)}\frac{d^+g_2}{ds}(0) = -\alpha\int_0^\infty n_+[l, \infty)\, e^{-\alpha l}\, dl \tag{2.27}$$

Substituting (2.24) into (2.27), we get

$$1/[h(\alpha)\alpha] = \int_0^\infty n_+[l, \infty)\, e^{-\alpha l}\, dl \tag{2.28}$$

Let us make a coordinate transformation. Put

$$\begin{gathered}\hat{e}_0(r) = e_0(s^{-1}(r)), \qquad \hat{e}_1(r) = e_1(s^{-1}(r)) \\ \hat{m}(r) = m(s^{-1}(r))\end{gathered} \tag{2.29}$$

Then on $(s(r_1), s(r_2))$ we have

$$\hat{e}_0(r) = 1 + \alpha\int_0^r\int_{0+}^{v+} \hat{e}_0(u)\, d\hat{m}(u)\, dv \tag{2.30}$$

$$\hat{e}_1(r) = r + \alpha\int_0^r\int_{0+}^{v+} \hat{e}_1(u)\, d\hat{m}(u)\, dv \tag{2.31}$$

$$h(\alpha) = \lim_{r\uparrow s(r_2)} \hat{e}_1(r)/\hat{e}_0(r) \tag{2.32}$$

The asymptotic form of $h(\alpha)$ appearing in (2.32) as $\alpha \to \infty$ is investigated by Kasahara.[11] The following lemma is a corollary to his result.

Lemma 2. For $C_8 > 0$ and $0 < q < 1$, the following two conditions are equivalent:

1. $h(\alpha) \sim C_8 D_q \alpha^{-q}$ as $\alpha \to \infty$
2. $\hat{m}(r) \sim C_8^{-1/q} r^{1/q - 1}$ as $r \to +0$

where

$$D_q = [q(1-q)]^{-q} \Gamma(1+q)\, \Gamma(1-q)^{-1}$$

Under the assumption of the theorem

$$\hat{m}(r) = m(s^{-1}(r)) \sim C_3 C_2^{-\mu/\lambda} r^{\mu/\lambda} \qquad \text{as} \quad r \downarrow 0$$

Applying Lemma 2 with $1/q - 1 = \mu/\lambda$, $C_8 = (C_3 C_2^{-\mu/\lambda})^{-q}$, i.e., $q = p$, $C_8 = C_2^{1-p} C_3^{-p}$, we have

$$h(\alpha) \sim C_2^{1-p} C_3^{-p} D_p \alpha^{-p} \qquad \text{as} \quad \alpha \to \infty \tag{2.33}$$

where p is given by (2.17). Here $1/[\alpha h(\alpha)]$ is the Laplace transform of $n_+[l, \infty)$ as given in (2.28), and has the asymptotic form

$$1/[\alpha h(\alpha)] \sim C_2^{p-1} C_3^p D_p^{-1} \alpha^{p-1} \qquad \text{as} \quad \alpha \to \infty$$

from (2.33). By virtue of the Tauberian theorem for the Laplace transformation,[12] we have the asymptotic form for n_+

$$n_+[l, \infty) \sim C_2^{p-1} C_3^p D_p^{-1} \Gamma(1-p)^{-1}\, l^{-p} \qquad \text{as} \quad l \downarrow 0 \tag{2.34}$$

Relation (2.34) together with Lemma 1 proves the theorem.

3. ORNSTEIN–UHLENBECK PROCESS AND A SELF-SIMILAR DIFFUSION PROCESS

As in Section 2, $\Delta = 0$ is assumed. The assumptions of the theorem are checked by evaluating the A_i, B_i in (2.9) and confirming (2.10), (2.11).

The Ornstein–Uhlenbeck process is given by a stochastic differential equation

$$d\xi(x) = b(\xi(x))\, dx + a(\xi(x))\, dB(x), \qquad \xi(0) = 0 \tag{3.1}$$

with

$$b(u) = -C_9 u \quad (C_9 > 0), \qquad a(u) = 1 \tag{3.2}$$

Here B is the one-dimensional Brownian motion. In this case the boundaries $r_1 = -\infty$, $r_2 = \infty$ are natural and ξ is persistent. Generally, for ξ given by (3.1), s and m take the form[10]

$$s(r) = \int_0^r du \exp[-H(u)] \tag{3.3}$$

$$m(r) = \int_0^r du \, 2a(u)^{-2} \exp[H(u)] \tag{3.4}$$

where

$$H(u) = \int_{u_0}^{u} 2b(v)\, a(v)^{-2}\, dv \tag{3.5}$$

We may take any point u_0 in $I = (r_1, r_2)$ as the lower bound in the integral (3.5). Change of u_0 only induces the transformation $s(r) \to C_{10} s(r)$, $m(r) \to C_{10}^{-1} m(r)$ $(C_{10} > 0)$.

In the case in which (3.2) holds, s and m have the asymptotic property (2.14) with $\lambda = \mu = 1$; hence (2.15) holds with $p = 1/2$. We note that λ, μ, and accordingly the exponent p of (2.17) do not change as long as the functions b and a satisfy

$$b(u) \to C_{11}, \qquad a(u) \to C_{12} \neq 0 \tag{3.6}$$

as $u \to 0$.

Let us next study the self-similar case (1.11). The boundaries $r_1 = -\infty$, $r_2 = \infty$ are both natural and ξ is persistent. The condition (1.10) with $\Delta = 0$ can be checked by ascertaining that the generators of $\xi(cx)$ and $c^H \xi(x)$ agree with each other; for any function f belonging to the domain of the generator A [Eq. (2.5)] of ξ, we have

$$\begin{aligned} \lim_{x \downarrow 0} \{E_r[f(\xi(cx))] - f(r)\}/x &= c \lim_{x \downarrow 0} \{E_r[f(\xi(cx))] - f(r)\}/cx \\ &= cAf(r) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \lim_{x \downarrow 0} \{E_r[f(c^H \xi(x))] - f(r)\}/x &= \lim_{x \downarrow 0} \{E_{r'}[h(\xi(x))] - h(r')\}/x \\ &= Ah(r') \end{aligned} \tag{3.8}$$

Here h is defined as $h(r) = f(c^H r)$, and $r' = c^{-H} r$. Since by the scaling property of s and m

$$s(r') = c^{-\lambda H} s(r), \qquad m(r') = c^{\lambda H - 1} m(r)$$

Eq. (3.8) becomes $cAf(r)$, agreeing with (3.7).

Now we apply the theorem with $\lambda = \lambda$, $\mu = -\lambda + 1/H$, and obtain

$$p = \lambda H \tag{3.9}$$

where $H > 0$ and $0 < \lambda < 1/H$. The above relation shows that p depends both on λ and H.

The corresponding stochastic differential equation to the self-similar process is obtained by applying (3.3)–(3.5) on $(0, \infty)$ and $(-\infty, 0)$ separately. We find

$$b(r) = \begin{cases} C_{13} r^{1-1/H}, & r > 0, \\ -C'_{13} |r|^{1-1/H}, & r < 0, \end{cases} \qquad a(r) = \begin{cases} C_{14} r^{1-1/(2H)}, & r > 0 \\ -C'_{14} |r|^{1-1/(2H)}, & r < 0 \end{cases} \tag{3.10}$$

Here

$$\begin{aligned} C_{13} &= (1-\lambda)/[C_5 C_6 \lambda(-\lambda + 1/H)] \\ C_{14} &= \{2/[C_5 C_6 \lambda(-\lambda + 1/H)]\}^{1/2} \end{aligned} \tag{3.11}$$

and similar expressions obtained by replacing C_5, C_6 by C'_5, C'_6 are valid for C'_{13}, C'_{14}. The relation (3.10) shows that our diffusion process has the same generator on $(0, \infty)$ as the self-similar diffusion process constructed by Lamperti[13] on $[0, \infty)$ or $(0, \infty)$.

A few remarks will be given on the stochastic differential equation (3.1) with coefficients (3.10). At $r = 0$, the functions a, b lack the Lipschitz continuity, and both become even singular if $H < 1/2$. It might be feared that this irregularity gives rise to a difficulty in constituting the process on the whole region $(-\infty, \infty)$. One way to overcome this difficulty is, as we have done here, to define the process in terms of the scale s and the speed measure dm. Applying (3.3)–(3.5) to (3.10) on $(-\infty, 0)$ and $(0, \infty)$ separately, we obtain s and m, naturally extensible to continuous functions on $(-\infty, \infty)$. The extended s and m, which essentially agree with (1.11), allow us to construct the process as follows. Take a Brownian motion $\{B(x), x \geqslant 0\}$ with $B(0) = 0$. Define the standard Brownian local time ϕ_B at r as [i.e., $dm(r) = 2dr$ in (2.13)]

$$\phi_B(x, r) = \text{measure}(y\colon B(y) \in dr, 0 \leqslant y \leqslant x)/2dr$$

and

$$\Phi(x) = \int_{-\infty}^{\infty} \phi_B(x, r)\, dm(s^{-1}(r))$$

Then the process $\xi(x) = s(B(\Phi^{-1}(x)))$ starts at 0 and has $d/dm\, d^+/ds$ as its generator.

The above construction is generally valid for any pair of a scale s and a speed measure dm; on $(-\infty, \infty)$, s is a continuous, strictly increasing function with $s(-\infty) = -\infty$, $s(\infty) = \infty$, and dm is a measure that is positive on each nonempty open set and is finite on each compact set. See Ref. 8 for further details.

Finally, we remark that, unlike the Ornstein–Uhlenbeck process, the self-similar process does not have a stationary version; the condition (Ref. 8, Section 4.11, problem 11)

$$m(r_1) > -\infty \qquad \text{and} \qquad m(r_2) < \infty$$

for the existence of an invariant probability measure (stationary probability function) is violated.

4. EXAMPLES NOT OBEYING POWER LAWS

In this section $N(l, L)$ is the same as in Section 2, but we do not set $\Delta = 0$. Take a Gaussian process ξ starting at Δ with mean Δ and covariance

$$E[\xi(x)\,\xi(y)] = xy + \Delta^2 \tag{4.1}$$

We see that the power law in Section 2 does not hold for ξ, since the total number of fractures is finite;

$$N(0+, L) < \infty \text{ with probability one} \tag{4.2}$$

for $L < \infty$. This comes from the fact that $\xi(x)$ is, with probability one, such an analytic function that its Taylor expansion at $x \in [0, \infty)$ has infinite radius of convergence.[14]

Suppose the contrary to (4.2); $\xi(x_i) = \Delta$ for infinitely many x_i in $(0, L)$ with positive probability. We may assume $0 < x_1 < x_2 < \cdots < L$. The case of $0 < \cdots < x_2 < x_1 < L$ can be proved similarly. Let x_∞ be the limit point of $\{x_1, x_2,...\}$. For $k = 0, 1,...,$ we can find a sequence $x_{k,1} < x_{k,2} < \cdots \uparrow x_\infty$ such that $\xi^{(k)}(x_{k,i}) = \Delta\delta_{k0}$. This is shown by induction. The case of $k = 0$ is clear by taking $x_{0,i} = x_i$. Assume $\{x_{n,i}\}$ has already been found. Since $\xi^{(n)}(x_{n,i}) = \xi^{(n)}(x_{n,i+1}) = 0$, by the mean value theorem there exists $x_{n+1,i} \in (x_{n,i}, x_{n,i+1})$ such that $\xi^{(n+1)}(x_{n+1,i}) = 0$. Here $x_{n+1,i}$ converges to x_∞, since $x_{n+1,i} \in (x_{n,i}, x_{n,i+1})$ and $x_{n,i}, x_{n,i+1} \to x_\infty$, and it is strictly increasing in i, since $x_{n+1,i-1} < x_{n,i} < x_{n+1,i}$. Hence the validity is shown in the case of $k = n+1$. The existence of the sequence $\{x_{k,i}\}$ implies that $\xi(x_\infty) = \Delta$, and that all the derivatives $\xi^{(k)}$ $(k = 1, 2,...)$ vanish at x_∞, since $\xi^{(k)}(x_\infty) =$

$\lim_i \xi^{(k)}(x_{k,i}) = \Delta\delta_{k0}$. By the analytic property of ξ, it is concluded that $\xi \equiv \Delta$ with positive probability. But this contradicts (4.1), since

$$\begin{aligned} &P(\xi(x) \equiv \Delta \text{ on } [0, L]) \\ &\quad \leqslant P(\xi(x_1) = \Delta) \\ &\quad = \int_{\{\Delta\}} (2\pi\sigma^2)^{-1/2} \exp[-(y-\Delta)^2/(2\sigma^2)]\, dy = 0 \\ &\quad (\sigma^2 = x_1^2 + \Delta^2) \end{aligned}$$

Next we discuss the stationary model proposed by Oda *et al.*[6] Let η be a stationary Gaussian process with mean 0 and covariance

$$E[\eta(x)\,\eta(y)] = \exp(-C_7\,|x-y|^2) \qquad (C_7 > 0) \tag{4.3}$$

The same analytic property of η as ξ makes the relation (4.2) hold, so that the power law in Section 2 is not valid either. Since η is ergodic, it has another interesting distribution $\bar{N}(l)$, the limit of the cumulative number of fractures per unit length $\lim_{L\to\infty} N(l, L)/L$. It seems difficult to express $\bar{N}(l)$ explicitly, but for sufficiently large Δ, it does not follow a power law:

$$\bar{N}(2l\Delta^{-1}C_7^{-1/2})/\bar{N}(0+) \to \exp(-l^2) \tag{4.4}$$

as $\Delta \to \infty$ (Ref. 15, Sections 11.5, 12.5).

The results (4.2), (4.4) remain valid under weaker conditions. See Ref. 15, Section 13.2 for further details.

5. PROCESSES WITH DISCONTINUOUS SAMPLE FUNCTIONS

The processes ξ and η in Sections 2–4 have continuous sample functions with probability one. Physically this corresponds to the situation that the processes represent the stress fields. Tsuboi,[16] on the other hand, proposed a fracture criterion of the form (1.2) in terms of a possibly discontinuous strain field ξ. Typical processes having discontinuous sample functions were investigated by Kesten.[17] Here we give a sketch of his result.

As in Section 2 we put $\Delta = 0$. Let ξ be the symmetric, stable process with index $1/H$ ($H \geqslant 1/2$), i.e., a stochastic process with stationary independent increments whose distribution is determined by (1.15). Essentially ξ is the Brownian motion if $H = 1/2$. The process ξ with $H > 1/2$ can be naturally regarded as a discontinuous counterpart of the Brownian motion; it is a process uniquely characterized by three properties, stationary independent increments, self-similarity (1.10) with $\Delta = 0$, and the invariance under $x \to -x$, i.e.,[19]

$$\xi(-x) \stackrel{\mathrm{d}}{=} \xi(x)$$

He calculated $N(l, L)$, the number of positivity intervals whose length is at least l. He found, roughly speaking, a power law

$$N(l, L) \sim l^{-p}, \qquad p = \max(1 - H, 0) \tag{5.1}$$

As in Section 2, the exponent p agrees with the Hausdorff dimension of $\xi^{-1}(0)$.[20] The precise implication of (5.1) is, however, different from the theorem in Section 2:

$$\lim_{l \downarrow 0} P(N(l, L)/f_H(l, L) \leqslant x) = \begin{cases} F_H(x), & 1/2 \leqslant H < 1 \\ G(x), & H = 1 \end{cases} \tag{5.2}$$

$$\operatorname*{l.i.p.}_{l \downarrow 0} N(l, L)/f_H(l, L) = 1, \qquad H > 1$$

Here l.i.p. means limit in probability, and the scaling function $f_H(l, L)$ is given by

$$f_H(l, L) = \begin{cases} H\Gamma(1-H)[\pi \sin(\pi H)]^{-1} (l/L)^{H-1}, & 1/2 \leqslant H < 1 \\ (2\pi^2)^{-1} [\log(L/l)]^2, & H = 1 \\ (2\pi)^{-1} H \tan(\pi/2H) \log(L/l), & H > 1 \end{cases} \tag{5.3}$$

The function $F_H(x)$ is the so-called Mittag–Leffler distribution

$$F_H(x) = \frac{1}{\pi(1-H)} \int_0^{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k!} \sin[\pi(1-H)k] \times \Gamma(1 + k(1-H))\, t^{k-1}\, dt$$

and

$$G(x) = \int_0^x \sum_{k=0}^{\infty} (-1)^k \pi(k + 1/2) \exp[-\pi^2(2k+1)^2 t/8]\, dt$$

A counterpart of the theorem in Section 2 seems to be unknown in the present case, although it has been established for the size distribution of zero free intervals.[18]

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